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Evolution of Hopf configuration of spin field in the Heisenberg model

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Abstract. A computer-algebra method for the analysis of evolution equations is presented. Results are given for the Hopf configuration linear-in-time approximation and the first-order nonlinear approximation. A hypothesis of asymptotic behaviour of the Hopf configuration is formulated. A test case of the Schrödinger equation and the so-called nonlinear Schrödinger equation is also presented.

1. Non-numerical approach to the evolution equation

In many fields of physics we use differential equations to describe the evolution of the system under consideration. These evolution equations often take the following form:

$$\frac{\partial f}{\partial t} = \hat{O} f . \tag{1}$$

When operator $\hat{O}$ is nonlinear, which is often the case, they are not easy to solve analytically and are usually treated numerically. Numerical integration of evolution equations is a well-developed method‡, but it needs discretized space to work with. In most situations this is not a problem. Problems arise only when we are looking for solutions which are topologically non-trivial. Discretization of space leads to problems with defining topological properties of a field under consideration, since they are defined for continous fields and are not easily extendable to discrete ones. Furthermore, numerical integration, due to its discrete nature, may not preserve topological charge during the evolution. Finally, numerical methods, in a multi-dimensional case, need substantial amounts of computing resources§, and are quite expensive in terms of computing time.

In many cases, we only need a clue to help us find a solution to the equation. In this case it is not necessary, and not worth investing time in the numerical approach.

Using the definition of a derivative, we may rewrite (1) in the approximate form

$$\frac{f_{t+\mathrm{d}t} - f_t}{\mathrm{d}t} = \hat{O} f_t \tag{2}$$

and then the form of the function f in the moment $t + \mathrm{d}t$ is described by the recursive formula

$$f_{t+\mathrm{d}t} = f_t + \mathrm{d}t \hat{O} f_t . \tag{3}$$

The above equation may, in principle, be iterated for any given starting field configuration f_0. Furthermore, field $f_{t+\mathrm{d}t}$ is homotopically equivalent to the field f_t.

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‡ For a summary of numerical methods see for instance [1, 2].

§ For example, one needs 128 megabytes of memory for a 256^3 lattice.

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Proof. By definition, loops $f(\sigma)$ and $g(\sigma)$ are homotopical in x_0 if there exists, continous in $\theta \in [0, 1]$, a function $h(\sigma, \theta)$ such that

$$h(\sigma, 0) = f(\sigma) \tag{4}$$

$$h(\sigma, 1) = g(\sigma) \tag{5}$$

$$h(\partial I, \theta) = x_0 . \tag{6}$$

It is obvious that if both f_t and f_{t+dt} satisfy the same boundary conditions, the function

$$h(\sigma, \theta) = f_t(\sigma) + \theta dt \hat{O} f_t(\sigma) \tag{7}$$

is our function for f_t and f_{t+dt}. □

Thus, we get *analytical* formulae for the field configuration for any given value of time.

In this paper we will show a method of non-numerical analysis of the first-order in time nonlinear evolution equations, using the evolution equation of the spin field (8) as an example. We will also present results for a configuration with non-trivial Hopf index, in the limit of a small value of the time parameter. Another two examples of the use of the above method to the well known cases of the Schrödinger equation and the nonlinear Schrödinger equation are given in appendix A. Finally we will formulate a qualitative hypothesis on the asymptotic behavior of the spin field in the limit of infinite time.

2. *Heisenberg ferromagnet with topological charge*

The non-singular configuration of the three-dimensional Heisenberg ferromagnet with topological charge has been discussed over the past two decades [3–6]. The Heisenberg model of a ferromagnet in the long-wave approximation is equivalent to the three-dimensional vector field with the length of vectors normalized to one. In other words it is a map,

$$s : \mathcal{R}^3 \mapsto S^2 \tag{8}$$

from three-dimensional Euclidian space to the two-dimensional sphere S^2. The energy of this field is expressed by the formula

$$H = \int_{\mathcal{R}^3} (\nabla s)^2 \, d^3x . \tag{9}$$

The field (8) has some associated topological properties, in particular we can classify all fields (8) using a *Hopf index* as a class number [7, 8]. The Hopf index for this field is defined as follows [9]:

$$q = -\int_{\mathcal{R}^3} d^3x A_\mu J^\mu \tag{10}$$

where

$$J^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} \epsilon_{abc} S^a \partial_\nu S^b \partial_\lambda S^c \tag{11}$$

and A satisfies

$$J^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda . \tag{12}$$

The integral (10) takes only integer values, and this property prevents any configuration from changing Hopf index by means of time evolution (as it is a continuous process). In [8] we have proved that this property is not sufficient to stabilize the configuration. This result leads to the natural question of how the configuration will evolve in time. This is considered in the next section.

3. Evolution of a Heisenberg ferromagnet

The time evolution of a spin field (8) in a Heisenberg ferromagnet without damping and anisotropy is governed by the following equation [10]:

$$\frac{\partial \omega}{\partial t} = \mathrm{i}\left(\Delta\omega - \frac{2\omega^*(\nabla\omega)^2}{1+\omega\omega^*}\right) \tag{13}$$

where ω is a stereographic coordinate of vector s defined as

$$\omega = \frac{s^x + \mathrm{i}s^y}{1+s^z}. \tag{14}$$

The coordinates of vector s are expressed in terms of variable ω by

$$s^x = \frac{2\mathrm{Re}(\omega)}{1+\omega\omega^*} \qquad s^y = \frac{2\mathrm{Im}(\omega)}{1+\omega\omega^*} \qquad s^z = \frac{1-\omega\omega^*}{1+\omega\omega^*}. \tag{15}$$

Equation (13) is highly nonlinear, and all attempts to solve it have failed so far. Furthermore, we are interested in solutions with non-trivial Hopf index. This requirement rules out a numerical approach to the problem as topological properties of the field s may not be preserved by numerical integration of the evolution equation.

Using the definition of the time derivative (13) becomes

$$\frac{\omega_{t+\Delta t} - \omega_t}{\Delta t} = \mathrm{i}\left(\Delta\omega - \frac{2\omega^*(\nabla\omega)^2}{1+\omega\omega^*}\right) \tag{16}$$

which can be rewritten as a recursive formula for the field configuration at the moment $t+\Delta t$,

$$\omega_{t+\Delta t} = \omega_t + \mathrm{i}\Delta t\left(\Delta\omega_t - \frac{2\omega_t^*(\nabla\omega_t)^2}{1+\omega_t\omega_t^*}\right). \tag{17}$$

Our idea was to iterate the above formula, for a given starting configuration, keeping all parameters, as well as the form of the function, analytical. As the result is correct in the limit of small evolution time we can learn something on how a given configuration evolves. We can also extrapolate our result to some finite time and, if the results from numerous steps are in agreement, see how the configuration evolves in the finite-time regime. This is very easy in principle. The problem is in the complexity of the formulae involved in this kind of computation. Until recent years this approach was not practical, as the formulae involved may be very long and complicated†. When, in the mid seventies, the first programs for symbolic calculations appeared, this kind of approach became possible. Only recently has progress in symbolic program development and in computer technology allowed us to actually use the above method as a practical tool. There are many programs on the market (Mathematica, Macsyma and Maple to name a few) which can be used to perform this kind of calculation.

An illustration of the method presented on the two examples is given in appendix A. In the first example we apply (3) to the time-dependent Schrödinger equation with a general, time-independent potential. The second example concerns the nonlinear Schrödinger equation.

† In one particular case the formula took some 95 MB of memory.

4. Results for the Hopf configuration

The simplest known configuration with non-trivial Hopf index is given by a map

$$\omega = \frac{u}{v} \tag{18}$$

where complex numbers u, v represent the stereographical coordinates of a point in the $\mathcal{R}^3$ space. Variables $u = \alpha + i\beta$ and $v = \gamma + i\delta$ are related to the cartesian coordinates on the $\mathcal{R}^3$ space by means of the stereographic projection

$$\begin{aligned} \alpha &= \frac{2x}{1+x^2+y^2+z^2} \qquad \beta = \frac{2y}{1+x^2+y^2+z^2} \\ \gamma &= \frac{2z}{1+x^2+y^2+z^2} \qquad \delta = \frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}. \end{aligned} \tag{19}$$

In cartesian coordinates on $\mathcal{R}^3$ space, (18) becomes

$$\omega_1 = \frac{2(y-ix)}{1-x^2-y^2-z^2-2iz}. \tag{20}$$

The first step of iteration (17) gives us the result

$$\omega_2 = \frac{2(y-ix)\left(1-x^2-y^2-z^2-2i(z+\Delta t)\right)}{(1-x^2-y^2-z^2-2iz)^2} = \omega_1\left(1+\frac{i\Delta t\omega_1}{y-ix}\right) \tag{21}$$

which, incidentally, is quite simple. This is not the case with further steps. Even the second step is too long to quote here and the third step exceeds the capacity of our computers. Furthermore, even for ω_3 we have problems with calculating the energy density of the configuration. Nevertheless, we managed to fully calculate two iteration steps of (17).

There is a problem with visualizing the properties of the field (8). We have developed several methods of representing this field pictorially. Unfortunately the pictures we get from these methods are so complicated that it is hard to see anything from them. One of our methods was to plot the spin vectors on a regular grid in space, but the resulting picture was completely illegible due to its complexity. Another attempt was to plot the field lines across space. Using known cylindrical symmetry of the field (20) to eliminate unnecessary lines we were able to produce images like figure 1 showing field lines passing around the origin of the coordinate system. However, this approach failed to produce meaningful images for more complicated cases, and thus we decided to use the energy density of (8) to visualize the evolution (figures 2). Figures 3 and 4 present plots of the energy density of the Hopf configuration in the x-z plane calculated for ω_2 and ω_3 respectively. As we can see they are in good agreement, and there is a toroidal† structure parting from the starting configuration and moving with approximately constant velocity‡.

Apparent agreement between the first and second steps of the calculation, depicted in figure 3 and figure 4 entitles us to use a large-time extrapolation as an aid to guessing the form of the evolution of the configuration. Figures 5 and 6 show how the field energy density may appear for $t = 200$. As we can see, the first and the second steps of the computation no longer agree but the disagreement is not complete. In the linear (in time)

† Energy density has cylindrical symmetry.

‡ We can make similar plots for different times and measure this velocity.

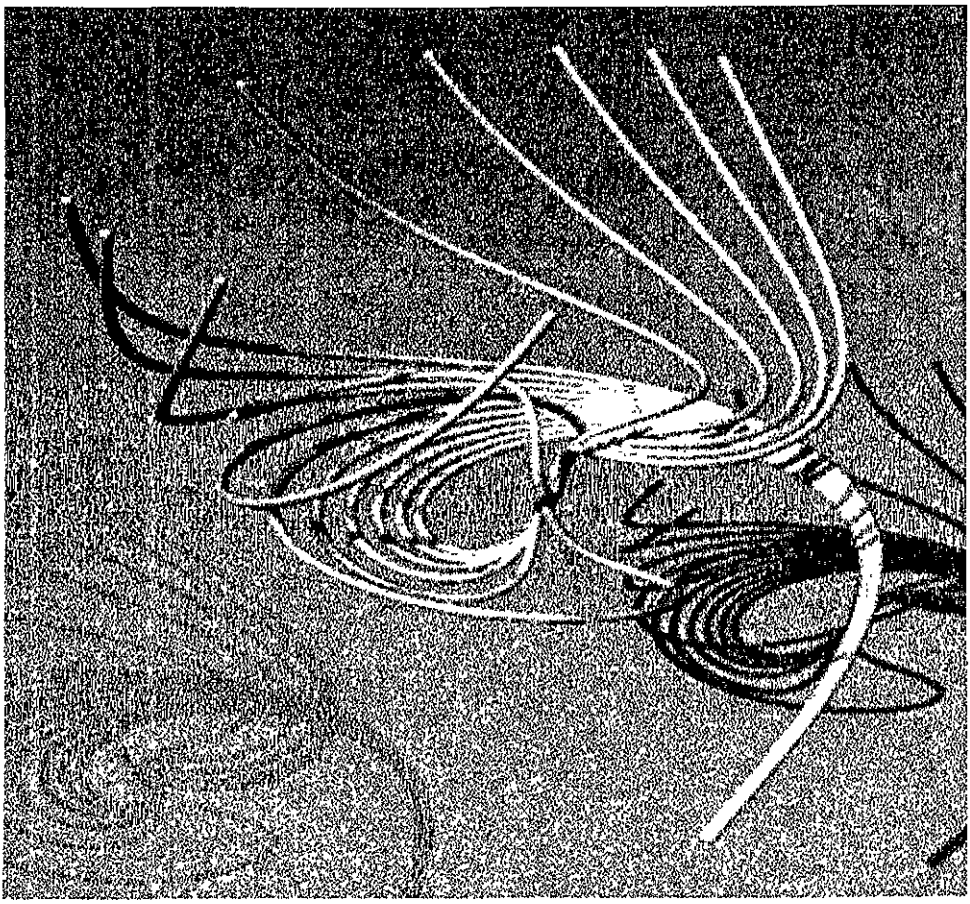

Figure 1. Field lines of the Hopf configuration of the spin field.

approximation we get two disjoint structures travelling with constant speed in opposite directions. In the nonlinear (in time) approximation this structure is spread on the surface of the growing sphere.

We feel that, while not being a proof or even a strong argument, this, together with the proof from [8], entitles us to state the hypothesis that the Hopf configuration (20) will expand to infinity in the limit of infinite time.

The somewhat similar problem of the Hopf configuration with the electromagnetic field has been studied in [11]. General properties of the solution presented there are similar to properties of our calculations. Both consist of structures with high energy density which spread in space.

All computations were performed on the Hewlett Packard Apollo 9000/710 workstation using Mathematica 2.1.

5. Conclusion

We have shown, as an alternative to numerical methods, an approach to the analysis of the evolution of particular configurations with non-trivial topological properties. This method

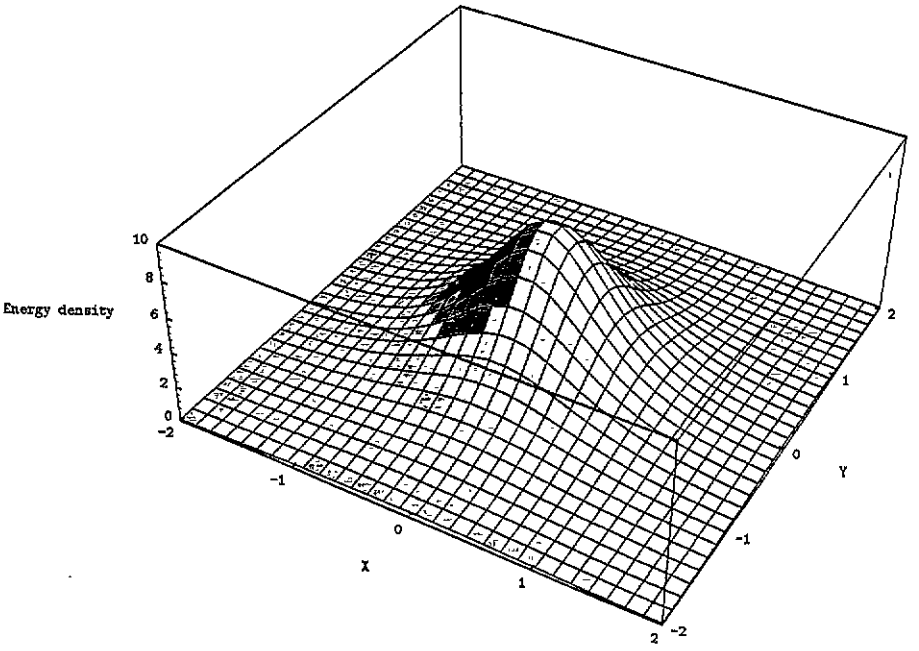

Figure 2. Energy density of the Hopf configuration for $t = 0$.

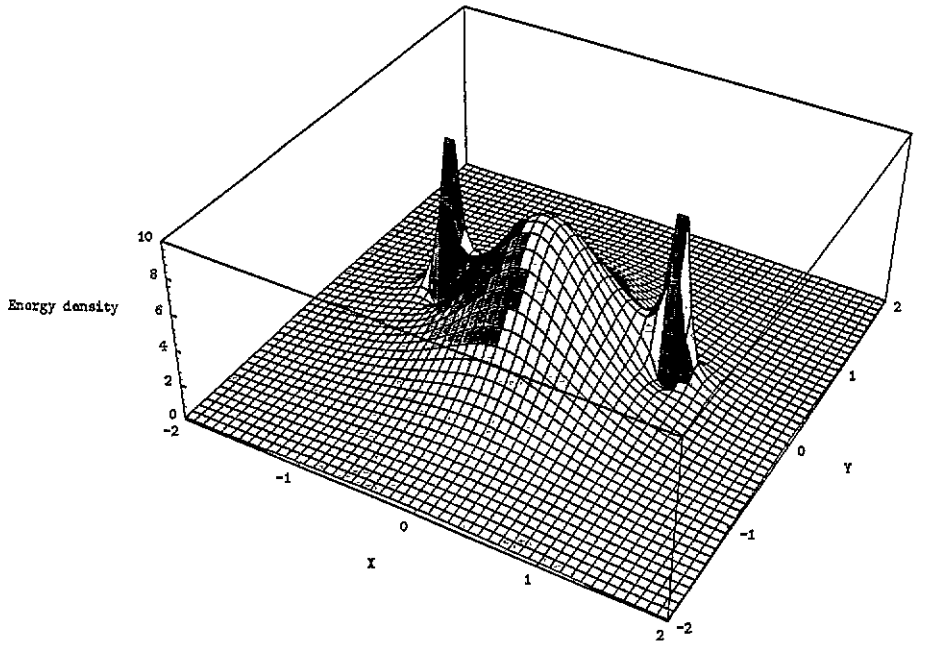

Figure 3. Energy density of the Hopf configuration for $t = 0.1$ for the first calculation step.

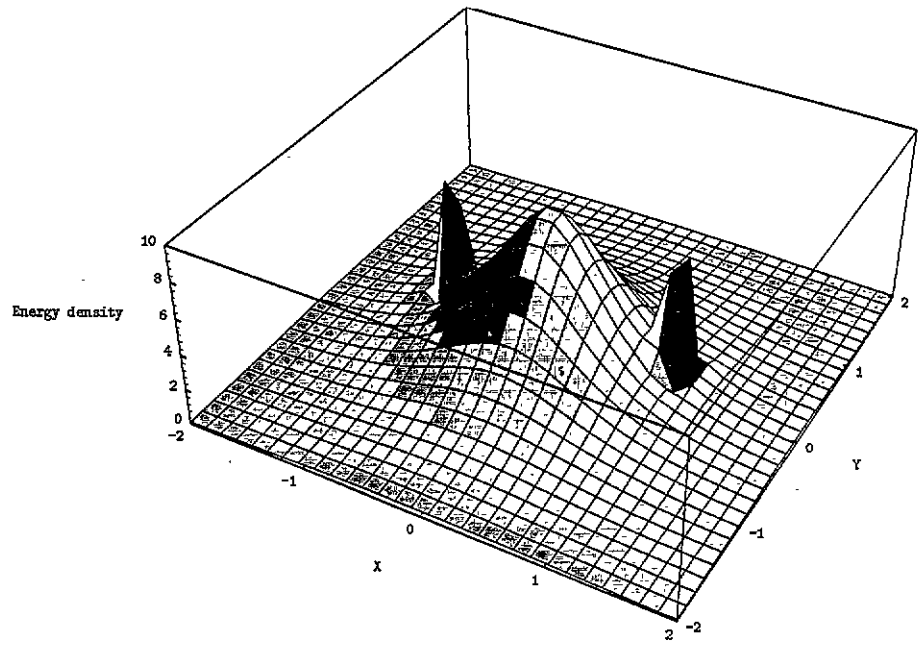

Figure 4. Energy density of the Hopf configuration for $t = 0.1$ for the second calculation step.

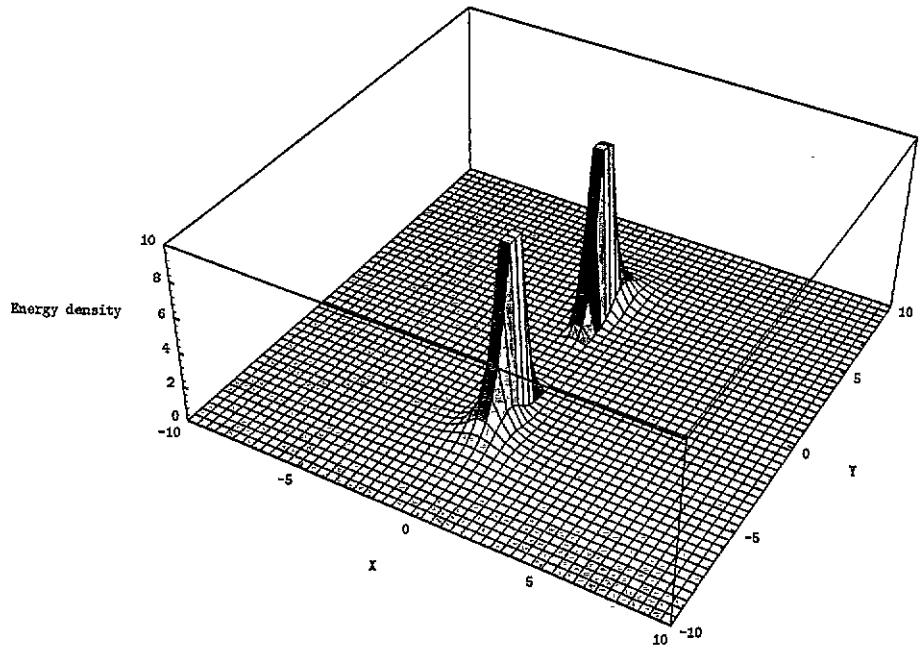

Figure 5. Energy density of the Hopf configuration for $t = 200$ for the first calculation step.

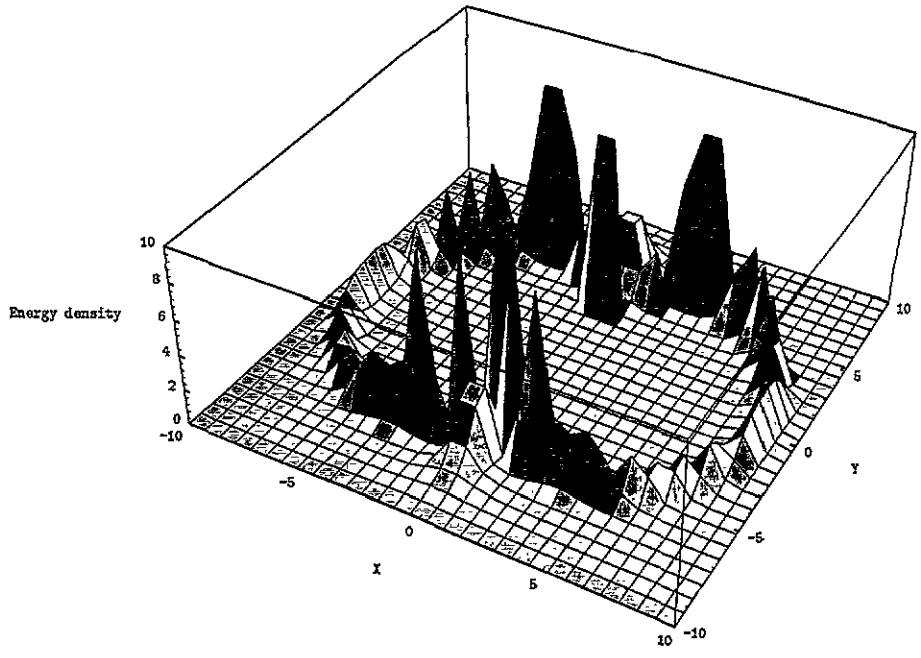

Figure 6. Energy density of the Hopf configuration for $t = 200$ for the second calculation step.

may be used for other, first-order in time, evolution equations. Results may be used, for instance, as a guide in the search for solutions or proper base states in the coherent states method.

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Appendix

In this appendix we will use the method presented above to solve the well known cases of the Schrödinger equation and the nonlinear Schrödinger equation.

Let us consider the linear Schrödinger equation in the form

$$\mathrm{i}\frac{\partial}{\partial t}\psi = (-\nabla^2 + V)\psi \tag{A1}$$

with an initial condition for the time $t = 0$ of

$$\psi(0, x) = \sum_n C_n \psi_n(x) \tag{A2}$$

where ψ_n

$$(-\nabla^2 + V)\psi_n = E_n \psi_n \tag{A3}$$

is a complete set of eigenfunctions of (A1). The equation equivalent to (17) for (A1) takes the form

$$\psi(\epsilon, \mathbf{x}) = \sum_n (1 - \mathrm{i}\epsilon E_n) C_n \psi_n(x) \tag{A4}$$

and for the Nth iteration step

$$\lim_{\epsilon = t/N;\, N\to\infty} \psi(N\epsilon, \mathbf{x}) = \lim_{\epsilon = t/N;\, N\to\infty} \sum_n (1 - \mathrm{i}\epsilon E_n)^N C_n \psi_n(x) = \sum_n C_n \mathrm{e}^{-\mathrm{i}tE_n} \psi_n(x) \tag{A5}$$

which is an *exact* solution of (A1) for any value of t and any initial condition (A2).

As a second example we will analyse a nonlinear Schrödinger equation

$$\frac{\partial \psi}{\partial t} = \mathrm{i}\left(\nabla^2 \psi + \alpha |\psi|^2 \psi\right) \tag{A6}$$

with an initial condition

$$\psi(0, x) = A\mathrm{e}^{\mathrm{i}k\cdot x}. \tag{A7}$$

The first iteration gives us

$$\psi(\epsilon, x) = A\left(1 + \mathrm{i}\epsilon(|A|^2\alpha - k^2)\right)\mathrm{e}^{\mathrm{i}k\cdot x} \tag{A8}$$

and the second iteration gives

$$\psi(2\epsilon, \mathbf{x}) = (1 - \mathrm{i}\epsilon\omega + \mathrm{i}\epsilon^3\alpha|A|^2\omega^2)(1 - \mathrm{i}\epsilon\omega)\mathrm{e}^{\mathrm{i}k\cdot x} \tag{A9}$$

where

$$\omega = \alpha|A|^2 - k^2. \tag{A10}$$

For small ϵ, (A9) approaches

$$\psi(2\epsilon, x) = A(1 - \mathrm{i}\epsilon\omega)^2\mathrm{e}^{\mathrm{i}k\cdot x} \tag{A11}$$

and thus

$$\psi(t, x) = \lim_{\epsilon = t/N; N\to\infty} \psi(N\epsilon, x) = \mathrm{e}^{\mathrm{i}(k\cdot x - \omega t)} \tag{A12}$$

which is the *exact* solution of (A6) with the initial condition (A7).

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